UNIVERSITY AT ALBANY
STATE UNIVERSITY OF NEW YORK

CSI 401 (Fall 2025)

Numerical Methods

Lecture 13: Nonlinear Equation Solver: Newton's Method

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Agenda

- One more nonlinear equation $F(x) = 0$ solver:
    - Newton's method
        - Variants: Quasi-Newton's method

Nonlinear equation solver: Newton's method

- Key idea:
  - Take F, find its local linear approximation at a starting point x_0, solve for x to get x_1, and use that as our new initial point.
  - Iterate until (hopefully) convergence.
- So, how to find the local linear approximation of F at x_0?
  - First-order Taylor expansion at x_0

$$P_1(x) = F(x_0) + F'(x_0)(x - x_0).$$

$$0 = F(x_0) + F'(x_0)(x - x_0) \implies -\frac{F(x_0)}{F'(x_0)} = x - x_0 \implies x = x_0 - \frac{F(x_0)}{F'(x_0)}.$$

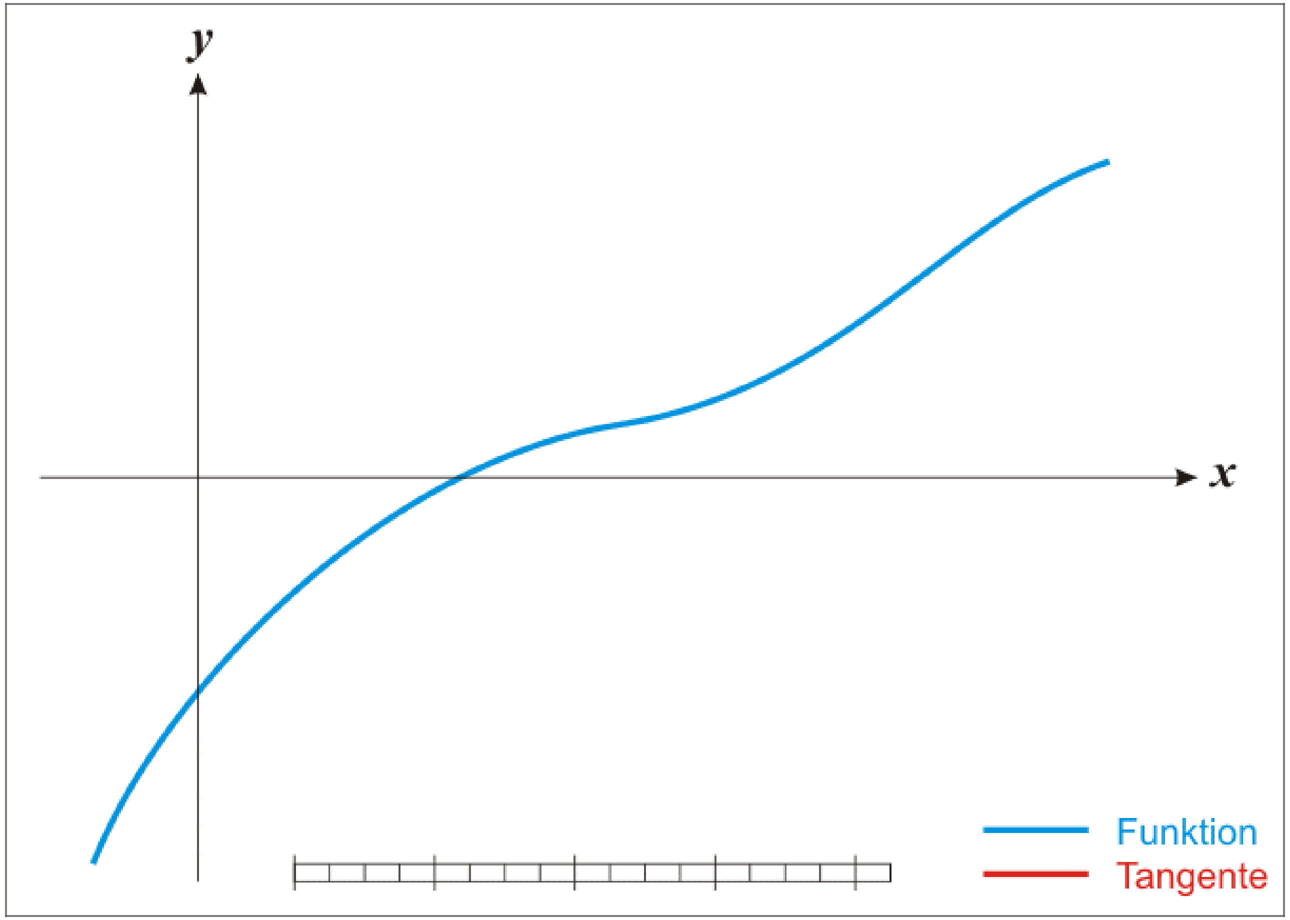
- Algorithm: (Newton update equation)

$$x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)}.$$

Illustration of the Newton’s method

- In each iteration:

$$x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)}.$$



Example of Newton's method

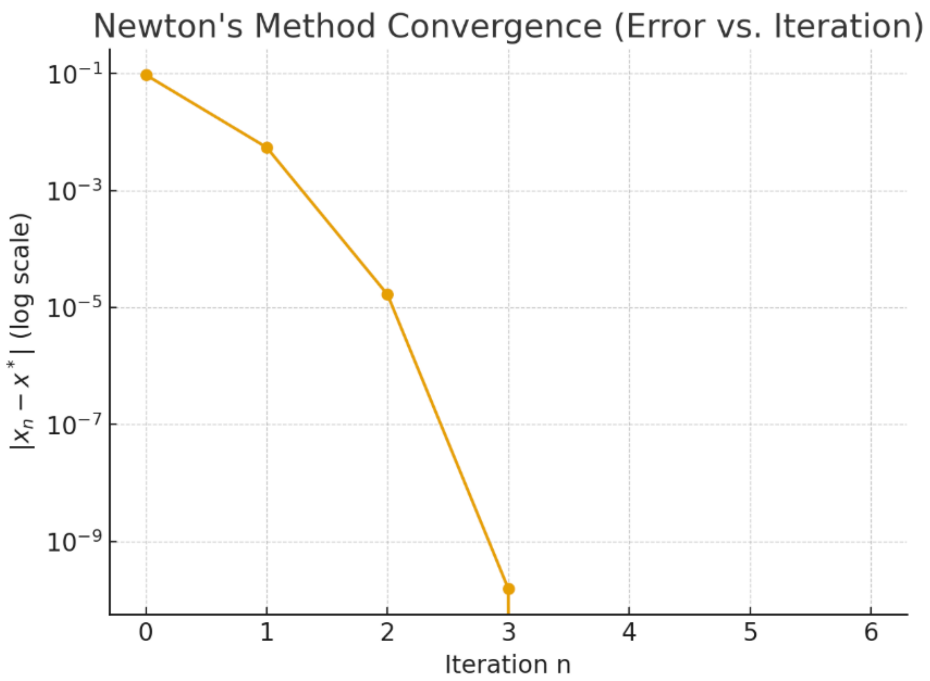
- Without a calculator, compute $x = \sqrt{2}$
  - Newton's method with 3 iterations
- Solutions:
  - Rewrite the problem as finding the root of $F(x) = x^2 - 2 = 0$
  - Then $F'(x) = 2x$
  - So updating rule is $x_{k+1} = x_k - \frac{x_k^2 - 2}{2x_k} = \frac{x_k}{2} + \frac{1}{x_k}$
    - Suppose $x_0 = 2$
    - $x_1 = 1 + \frac{1}{2} = 1.5$
    - $x_2 = 0.75 + \frac{1}{1.5} = 1.41666667 \ldots$
    - $x_3 = 1.41421569\ldots$
- Check $x_3^2 = 2.00000602$

In-class exercise – calculators needed

- Use Newton's method to find a root of
  - $f(x) = x^3 - 2x - 5 = 0$
  - With $x_0 = 2$ and 3 iterations.

- Solutions

Iteration (n)	x_n	f(x_n)	f'(x_n)	x_(n+1)	|x_(n+1) - x_n|
0	2.0000	-1.0000	10.0000	2.1000	0.1000
1	2.1000	0.0610	11.2300	2.0946	0.0054
2	2.0946	-0.0010	11.1580	2.0947	0.0001
3	2.0947	0.0000	11.1580	—	—



In-class exercise – Newton's method

- Solve $F(x) = x^3 - 2x + 2 = 0$
  - $x_0 = 0$ with 2 iterations.

$$x_{k+1} = x_k - \frac{x_k^3 - 2x_k + 2}{3x^2 - 2}.$$

$$x_1 = x_0 - \frac{2}{-2} = 1,$$

$$x_2 = 1 - \frac{1 - 2 + 2}{3 - 2} = 1 - 1 = 0 = x_0.$$

- Solve $F(x) = x^{1/3} = 0$
  - $x_0 = 2$ with 2 iterations.

$$x_{k+1} = x_k - \frac{x_k^{1/3}}{1/3 \cdot x^{-2/3}} = x_k - 3x_k = -2x_k.$$

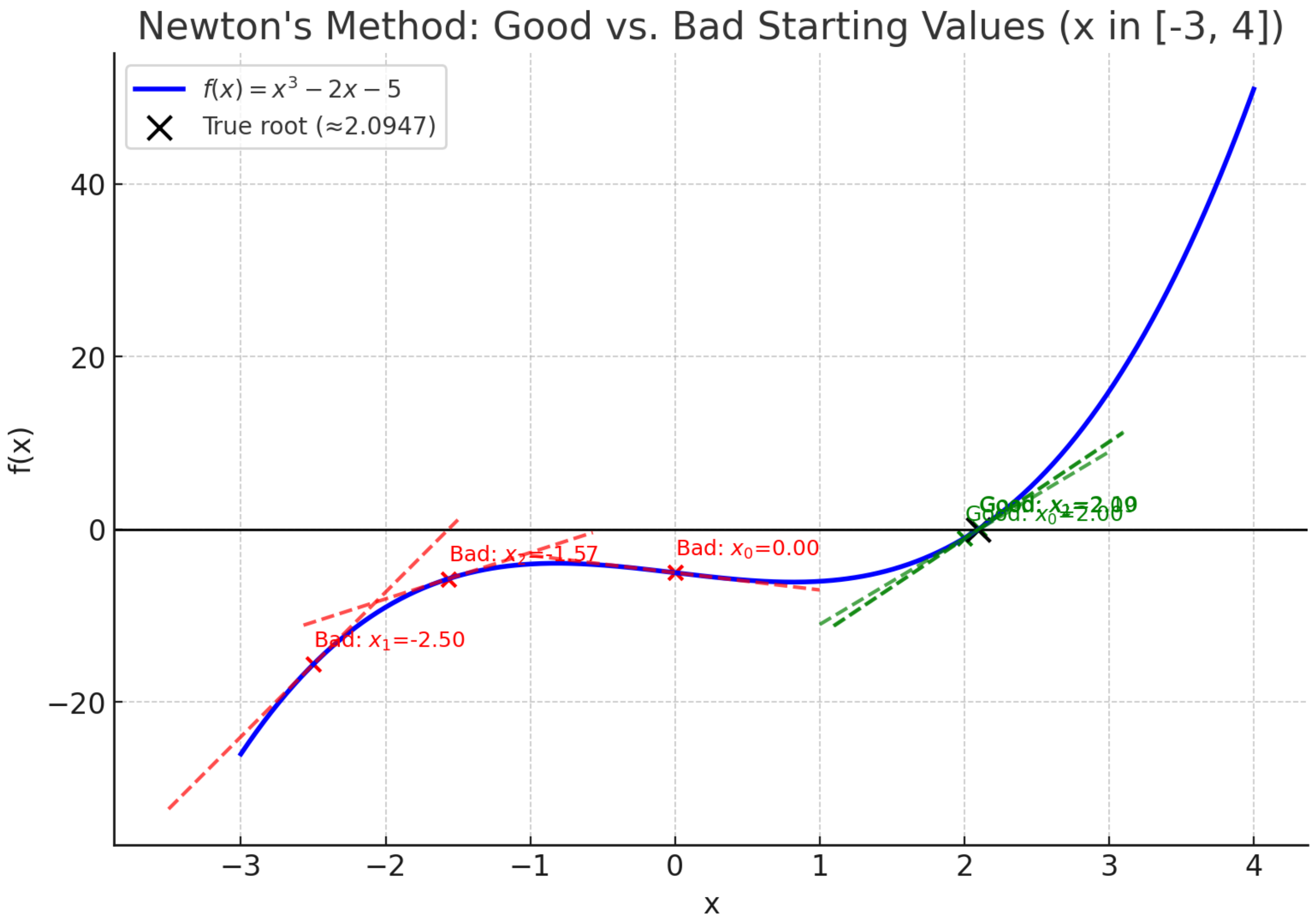
- What did you find?

Convergence of Newton's method

- Newton's method doesn't always converge to the root!

- **Theorem:**
  - Newton's method converges quadratically if:
    1. $f(x)$ is continuously differentiable near r,
    2. $f'(r) \neq 0$,
    3. The starting value x_0 is sufficiently close to r.
  - If $f'(r) = 0$ or x_0 is far from the root, convergence may be slow, linear, or divergent.

Comparison of good/bad starting points



Variants of Newton’s method: Quasi-Newton methods

- Sometimes, derivatives are expensive to compute, or we can’t compute them explicitly

- Constant slope method
  - Assume derivative is constant in the intervening iterations.
    - E.g., update the the calculation of the derivative every tenth iteration.

- Secant method
  - Start with two initial points x_0, x_1
  - Updating rule:

$$x_{k+1} = x_k - \frac{F(x_k)}{\frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}}}.$$

Summary of nonlinear equation solvers

- Things to know about:
  - Problem statement
  - Assumptions behind each method
  - Benefits/drawbacks of each method
  - Key theorems from calculus that feature in their analysis
  - How does each method look, visually?
  - How do we code each method up in Matlab/Python?
- Technical summary table:

Methods	Bisection method	Newton's method
Assumptions	Continuity, opposite sign condition	Continuous, differentiable, initial point close to root
Associated theorem	Intermediate value theorem	Taylor's remainder theorem
Guarantee	Linear convergence	Quadratic convergence