UNIVERSITY AT ALBANY
STATE UNIVERSITY OF NEW YORK

CSI 401 (Fall 2025)

Numerical Methods

Lecture 11: Optimization: Linear Programming

Chong Liu

Department of Computer Science

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What's Linear Programming (LP)?

- An optimization problem of linear objective functions with linear constraints.
    - Objective function can be minimized or maximized
    - Constraints can be in equalities or inequalities
    - All functions must be linear functions

- 2 examples:

$$\begin{aligned} \min\ & x_1 + 2x_2 \\ \text{s.t. } & x_1 + x_2 \leq 3 \\ & x_1 \geq 1 \end{aligned}$$

$$\begin{aligned} \max\ & x_1 + 2x_2 \\ \text{s.t. } & x_1 + x_2 = 3 \end{aligned}$$

- Discussion: Could you propose more linear programming problems?

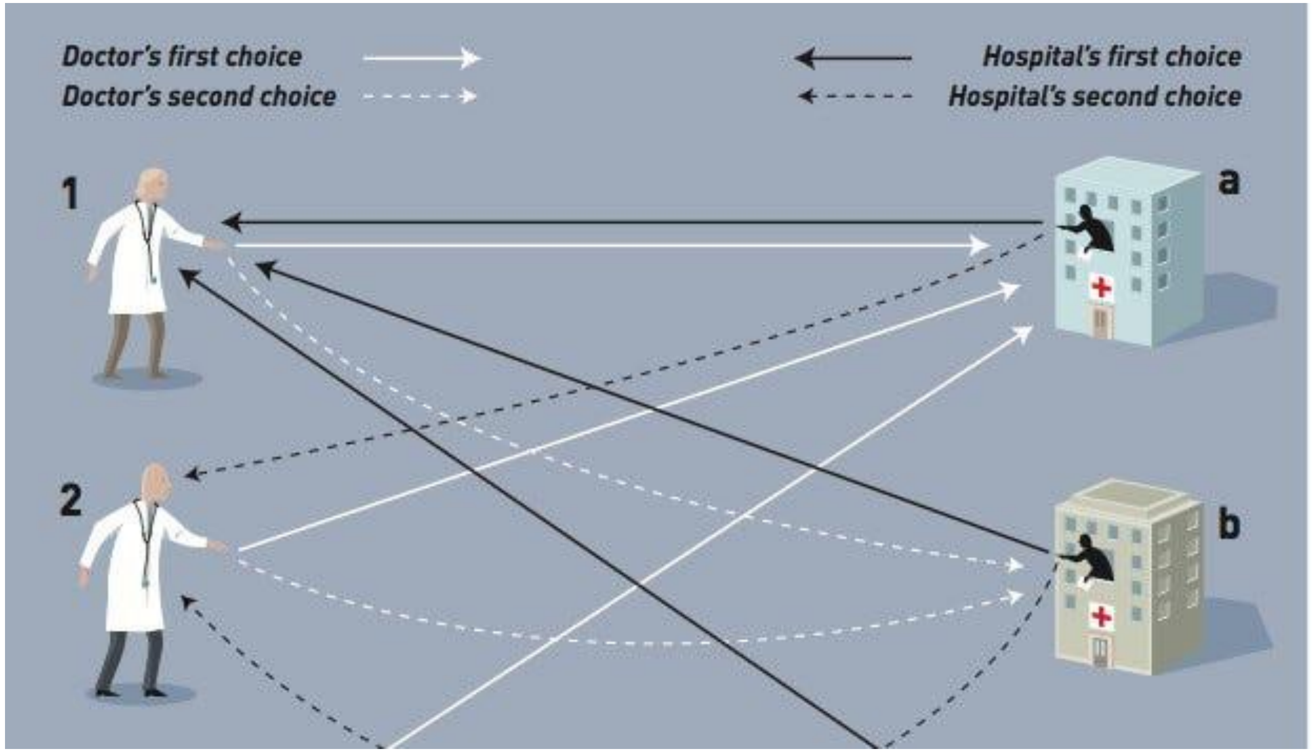
Application of LP: Matching problem

- Company (hospital) - Candidate (doctor) matching problem

- Each doctor:
  - Fits one position

- Doctors/hospitals:
  - Have their preferences

- Goal:
  - Put doctors to positions
  - Such that overall best match



Application of LP: Matching problem

- Company (hospital) – Candidate (doctor) matching problem

- Suppose now we have 4 doctors and 4 hospitals, preferences are listed in descending order in two ways:

    - Alice: Hospital A, Hospital B, Hospital C, Hospital D
    - Bob: Hospital B, Hospital C, Hospital A, Hospital D
    - Claire: Hospital C, Hospital D, Hospital A, Hospital B
    - David: Hospital D, Hospital A, Hospital B, Hospital C

    - Hospital A: David, Alice, Bob, Claire
    - Hospital B: Bob, Claire, Alice, David
    - Hospital C: Claire, David, Bob, Alice
    - Hospital D: Alice, David, Claire, Bob

Application of LP: Matching problem

- Company (hospital) – Candidate (doctor) matching problem
    - $x_{ij} \in \{0,1\}$ denotes assignment of doctor i to hospital j
    - $c_{ij} \in [0, 1]$ denotes the match between doctor i and hospital j
  - The objective function maximizes the overall match
  - 1st constraint ensures each hospital can hire a doctor
  - 2nd constraint ensures each doctor can find a job

$$\max \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} x_{ij}$$

$$\text{s.t. } \sum_{i=1}^{N} x_{ij} = 1, \ \forall j = 1, 2, \ldots, N$$

$$\sum_{j=1}^{N} x_{ij} = 1, \ \forall i = 1, 2, \ldots, N$$

Application of LP: Optimal transport

- Suppose you run a company, which has 4 factories and 3 big markets, each in a different city.

Boston demands 25

Buffalo supplies 15

Syracuse supplies 15

Rochester supplies 20

Albany supplies 35

New York City demands 30

Philadelphia demands 30

- You job is to design the optimal transportation route that has minimum transportation cost of your products
  - Each route (supply to demand) costs differently
  - Each factory has its supply capacity
  - Each market must be well supplied to maximize your profit

Application of LP: Optimal transport

- p_i is the supply capacity of each factory
- d_j is the demand of each market
- x_{ij} is the amount of products that are transported from factory i to market j
- c_{ij} is the transportation cost per product from factory i to market j

- Problem setup:

$$\min \sum_{i=1}^{N} \sum_{j=1}^{M} c_{ij} x_{ij}$$

$$\text{s.t. } \sum_{i=1}^{N} x_{ij} = d_j, \ \forall j = 1, 2, \ldots, M$$

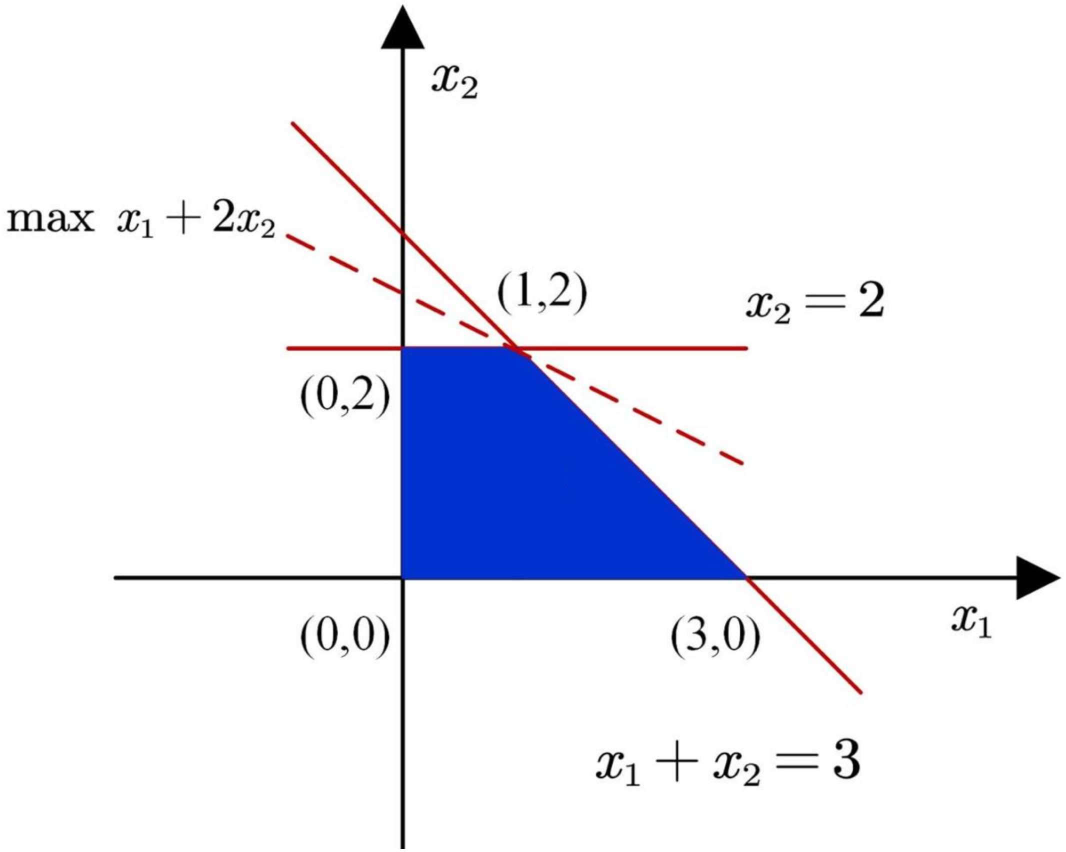
$$\sum_{j=1}^{M} x_{ij} \leq p_i, \ \forall i = 1, 2, \ldots, N$$

Applications of LP: Almost everywhere

- Resource allocation
- Transportation and logistics
- Portfolio optimization
- Manufacturing and production scheduling

How to solve the LP problem?

$$
\begin{aligned}
\text{maximize } x_1 + 2x_2 & \\
\text{subject to } x_1 + x_2 &\le 3 \\
x_2 &\le 2 \\
x_1 &\ge 0 \\
x_2 &\ge 0
\end{aligned}
$$



- For most 2-d LP problems,
  1. We can draw it's feasible region
  2. And move it's objective function

- In-class exercise: Draw the feasible region defined by constraints.

How to solve the LP problem?

- In-class exercise
  - Max profit in product planning

$$\text{Maximize } Z = 5x_1 + 4x_2$$

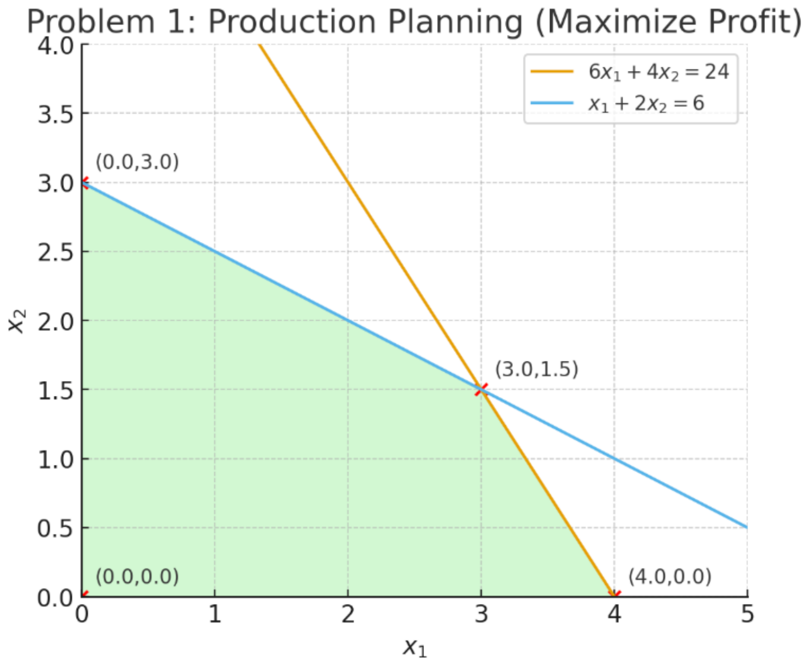
$$6x_1 + 4x_2 \leq 24 \quad \text{(Machine hours)}$$

$$x_1 + 2x_2 \leq 6 \qquad \text{(Labor hours)}$$

$$x_1, x_2 \geq 0$$

How to solve the LP problem?

- In-class exercise
  - Max profit in product planning



How to solve the LP problem?

- In-class exercise
  - Dietary cost optimization

$$\text{Minimize } C = 3x_1 + 2x_2$$

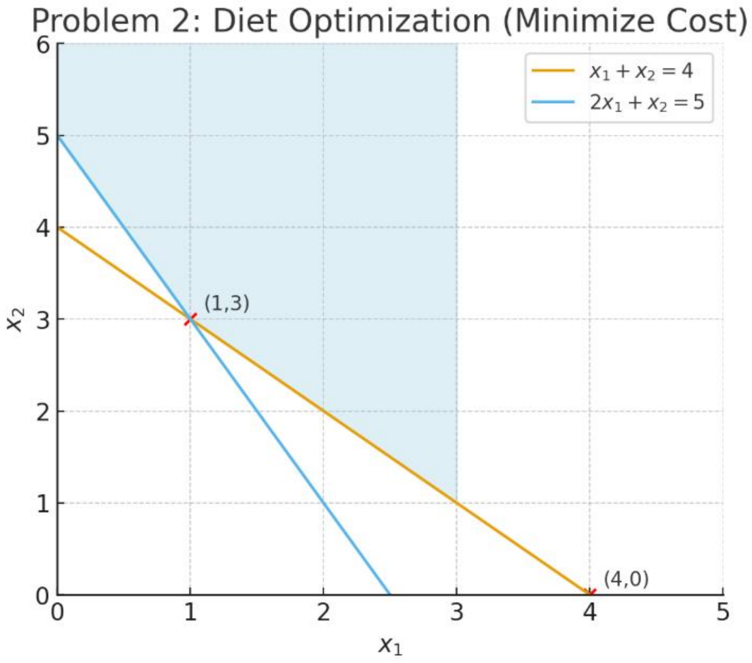
$$x_1 + x_2 \geq 4 \quad \text{(Protein requirement)}$$

$$2x_1 + x_2 \geq 5 \quad \text{(Vitamin requirement)}$$

$$x_1, x_2 \geq 0$$

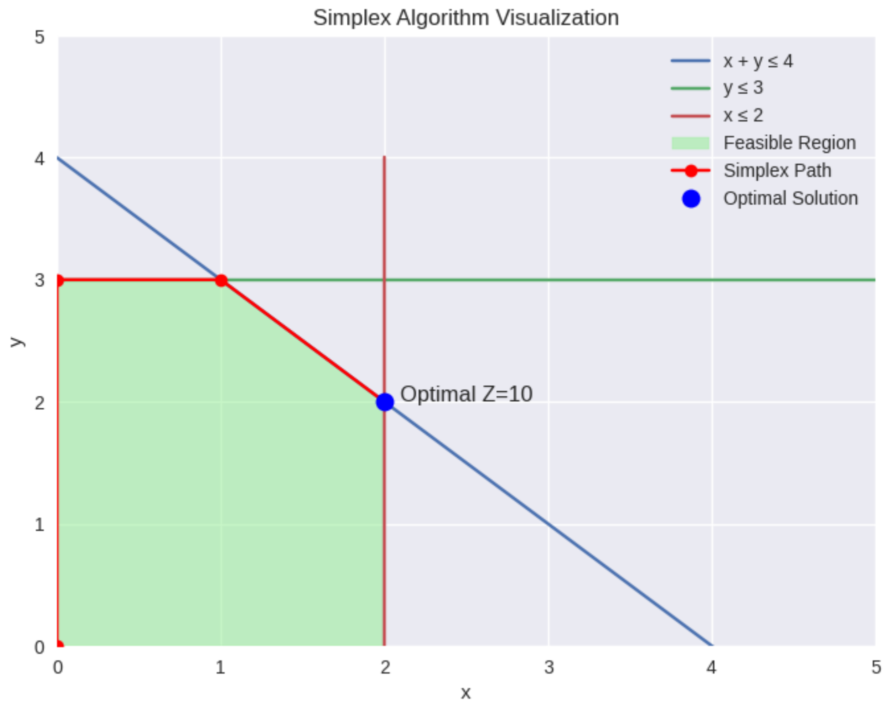
How to solve the LP problem?

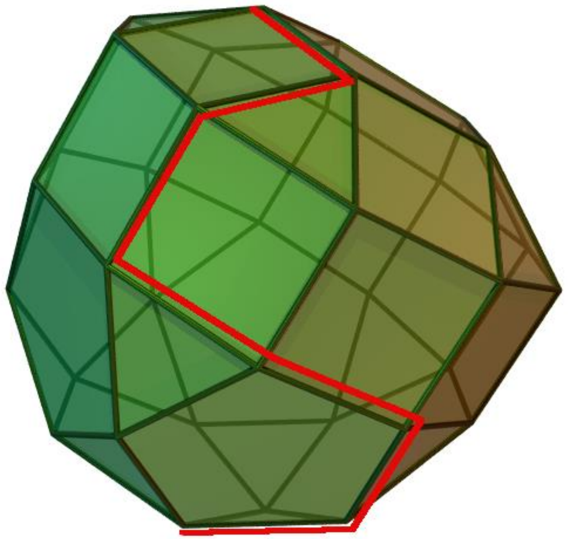
- In-class exercise
  - Dietary cost optimization



More algorithms for LP

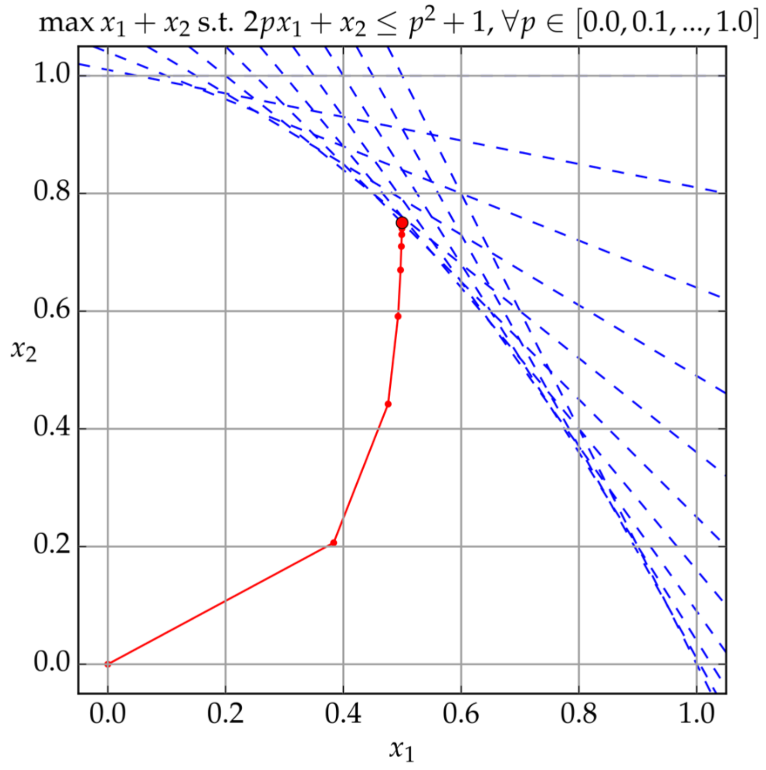
- Simplex method
  - Proposed by George Dantzig
    - During WWII
  - Key idea: since all optimal solutions sit at the edge of convex feasible region, why don't we search over the edge?
  - Pro: Efficient when number of variables is small
  - Con: Exponential time complexity in worst case
  - Python: scipy.optimize.linprog(method='simplex')
  - Matlab: linprog()





More algorithms for LP

- Interior point method
  - Key idea: start from a point in the feasible region, iteratively move towards the optimal solution.
  - Discussion: it reminds you which algorithm that we just learned in Lecture 10?
  - Pros: Can handle large number of variables in LP
  - Cons: Convergence is sensitive to choices of parameters



Dual problem of linear programming

- For every **primal** linear program, there is an associated **dual** LP that expresses the same optimization problem from a different perspective.

- Primal LP:
  - Max $c^T x$ $s.t.$ $Ax \leq b,\ x \geq 0,$
- Dual LP:
  - Min $b^T y$ $s.t.$ $A^T y \geq c,\ y \geq 0.$

- They are mathematically linked — this is not coincidence, but a property of convex optimization and linear algebra.

Dual problem of linear programming

- Economic Interpretation
  - The dual variables y represent **shadow prices** — the value of relaxing each constraint by one unit.
  - In a resource allocation problem, each y_i tells how much the objective (profit) would improve if resource i were increased slightly.

  - **Primal:**
    Max $3x_1 + 2x_2$
    s.t. $x_1 + 2x_2 \leq 8, 4x_1 + 3x_2 \leq 12, x_1, x_2 \geq 0$
  - **Dual:**
    Minimize $8y_1 + 12y_2$
    s.t. $y_1 + 4y_2 \geq 3, 2y_1 + 3y_2 \geq 2, y_1, y_2 \geq 0$

Duality Theorems

- Weak Duality:
  For any feasible x (primal) and y (dual), $c^T x \leq b^T y$.
  - The dual provides an upper bound (for maximization problems).

- Strong Duality:
  At the optimal solutions x^*, y^*, $c^T x^* = b^T y^*$.
  - Solving one problem solves the other — they share the same optimal value.

Dual problem of linear programming

- Why we study dual problems?

- Duality helps:
  - **Check optimality:** If primal and dual feasible solutions give the same objective, both are optimal.
  - **Perform sensitivity analysis:** Dual variables show how changes in constraints affect the outcome.
  - **Simplify computation:** Some LPs are easier to solve in dual form (e.g., when constraints >> variables).