UNIVERSITY AT ALBANY
STATE UNIVERSITY OF NEW YORK

CSI 401 (Fall 2025)

Numerical Methods

Lecture 8: Conditions of Linear Systems

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Announcements

- HW2 due tonight

- Midterm exam next Monday (2nd session in class)
  - Covers Lecture 1-7
    - HW1 (L1-3) HW2 (L4-7) and in-class exercise/discussion questions are your best friend!
  - Given individually
  - Given in closed-book
    - No lecture slides, notes, phones, cheatsheets, ...
    - All you need is a pen
      - Calculator is allowed, but you might not need it.
  - Counts 20 points towards your final grade

Recap: Eigenvalues and eigenvectors

- Definition:
  - For an $n \times n$ matrix A, an eigenvector v of A is a nonzero vector such that there exists some $\lambda \in R$ satisfying
  - $Av = \lambda v.$

- Work with the characteristic equation for the eigenvalues λ:

$$Av = \lambda v \Leftrightarrow Av - \lambda v = 0 \Leftrightarrow Av - \lambda Iv = 0 \Leftrightarrow (A - \lambda I)v = 0.$$

- But v is a nonzero vector, so $\det(A - \lambda I) = 0.$

Recap: Power method: Computing the eigenvalue of largest modulus and its corresponding eigenvector

$$|\lambda_1| > |\lambda_2| \geqslant \ldots \geqslant |\lambda_n|.$$

- Works for diagonalizable matrix only. All symmetric matrices are diagonalizable.

- Algorithm:
  - Start with an initial nonzero vector $w^{(0)}$
  - Run in K iterations

$$w^{(k+1)} = \frac{Aw^{(k)}}{\|Aw^{(k)}\|_2}.$$

  - Then your final $w^{(K)} \approx v_1$
  - And $\lambda_1 = Av_1/v_1$

Recap: Jacobi method

- After DLU decomposition, we have
  - $(D + L + U)x = b$
- Rearranging gives:
  - $Dx = b - (L + U)x$
- Jacobi iteration updates:
  - $x^{(k+1)} = D^{-1}(b - (L + U)x^{(k)})$.
  - Or equivalently:
    - $x^{(k+1)} = D^{-1}b + \underbrace{(-D^{-1}(L + U))}_{=:\,T_J} x^{(k)}$.
- So in compact form:
  - $x^{(k+1)} = T_J x^{(k)} + c$, *where* $c = D^{-1}b$.

Recap: Gauss-Seidel method

- After DLU decomposition, we have
  - $(D + L + U)x = b$
- Rearranging gives:
  - $(D + L)x = b - Ux$
- Gass-Seidel iteration updates:
  - $(D + L)x^{(k+1)} = b - Ux^{(k)}$
  - Formally, $x^{(k+1)} = T_{GS}x^{(k)} + c_{GS}$
    - where $T_{GS} = -(D + L)^{-1}U, c_{GS} = (D + L)^{-1}b$

Agenda

- Matrix rank
- Conditions of linear system solutions
- Geometric view of conditions

Matrix rank of A ($m \times n$ matrix)

- Definition:
  - Maximal number of linearly independent columns or maximal number of linearly independent rows.

- Two examples: Find ranks using Gaussian Elimination.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

- Rank is number of pivots after Gaussian Elimination.

In-class exercise

- Find the rank of this matrix.

$$A = \begin{bmatrix} 2 & 4 & 1 & 3 \\ 1 & 2 & 1 & 1 \\ 3 & 6 & 2 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Properties of matrix rank of A ($m \times n$ matrix)

- Rank doesn't change after elementary row operations.

- $0 \leq rank(A) \leq \min\{m, n\}$
- $rank(A) = rank(A^{\top})$

- Full column rank: $rank(A) = n$
- Full row rank: $rank(A) = m$

- For square A, $rank(A) = n \leftrightarrow \det(A) \neq 0$

Recap: Linear systems

- An example of linear systems
  - Any linear system can always be rewritten in matrix form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- More generally, $Ax = b$
  - A is an $m \times n$ matrix
  - x is an n-dimensional vector
  - b is an m-dimensional vector
- Problem: given A and b, how can you solve x?

In-class exercise: Gaussian Elimination

- Solve the following linear system:

$$\begin{cases} x + y + z = 1 \\ 2x + 2y + 2z = 2 \\ x + y + z = 3 \end{cases} \iff \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{array}\right]$$

- Solution:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 3 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

- What did you find?

In-class exercise: Matrix rank

- Find $rank(A)$ and $rank(A|b)$

$$\begin{cases} x + y + z = 1 \\ 2x + 2y + 2z = 2 \\ x + y + z = 3 \end{cases} \iff \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{array}\right]$$

- Solution:
  - $rank(A) = 1, rank(A|b) = 2$

Conditions of no solution to a linear system

- $rank(A) < rank(A|b)$

- In other words, the system is *inconsistent*.

In-class exercise: Gaussian Elimination

- Solve the following linear system:

$$\begin{cases} x + y + z = 1 \\ 2x + 2y + 2z = 2 \\ x - y + z = 0 \end{cases} \iff \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

- Solution:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- What did you find?

In-class exercise: Matrix rank

- Find $rank(A)$ and $rank(A|b)$

$$\begin{cases} x + y + z = 1 \\ 2x + 2y + 2z = 2 \\ x - y + z = 0 \end{cases} \iff \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

- Solution:
  - $rank(A) = rank(A|b) = 2$

Conditions of infinitely many solutions to a linear system

- Consistency (at least one solution exists):
  - $rank(A) = rank(A|b)$
- AND
- Underdetermined (free variables remain):
  - $rank(A) < n$

Parametric solution to a system with infinitely many solutions

- Solve the following linear system:

$$\begin{cases} x + y + z = 1 \\ 2x + 2y + 2z = 2 \\ x - y + z = 0 \end{cases} \iff \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

- Parametric solution: $x + y = 1 - z, \qquad x - y = -z.$

$$x = \tfrac{1-2t}{2}, \qquad y = \tfrac{1}{2}, \qquad z = t.$$

$$\left\{ \left(\tfrac{1-2t}{2},\ \tfrac{1}{2},\ t \right) :\ t \in \mathbb{R} \right\}.$$

Conditions of one unique solution to a linear system

- Consistency (at least one solution exists):
  - $rank(A) = rank(A|b)$
- AND
- Full rank (no free variables):
  - $rank(A) = n$

- When $m = n$ (square matrix)
  - If A is nonsingular ($\det(A) \neq 0$), then $x = A^{-1}b$ is the unique solution.
  - If $\det(A) = 0$, uniqueness fails (either no solution or infinitely many solutions).

In-class exercise: Geometric view

- Draw these two lines in the x-y coordinate space of the following linear system:

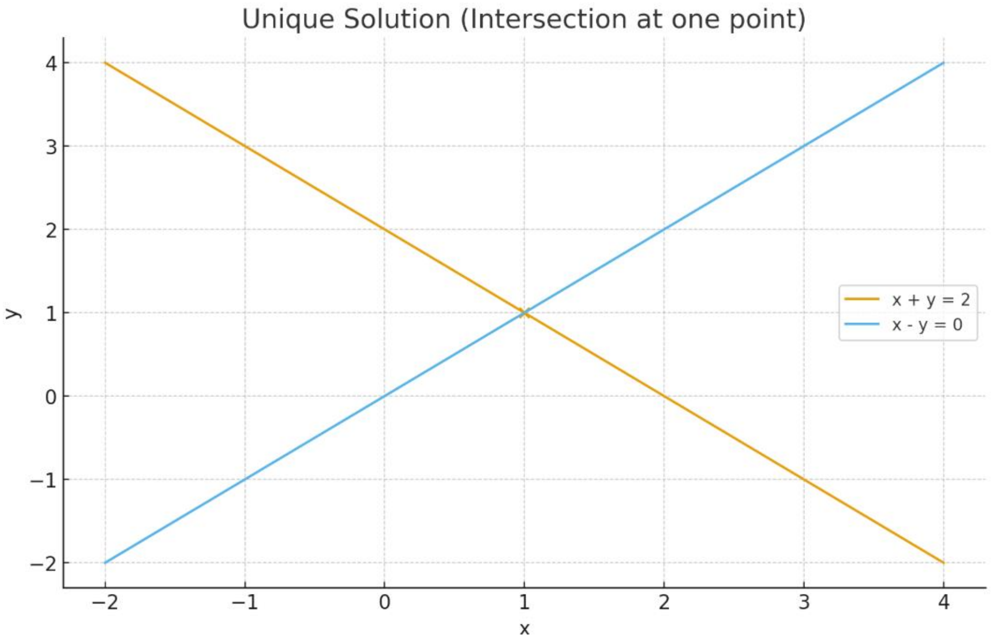
$$\begin{cases} x + y = 2 \\ x - y = 0 \end{cases}$$

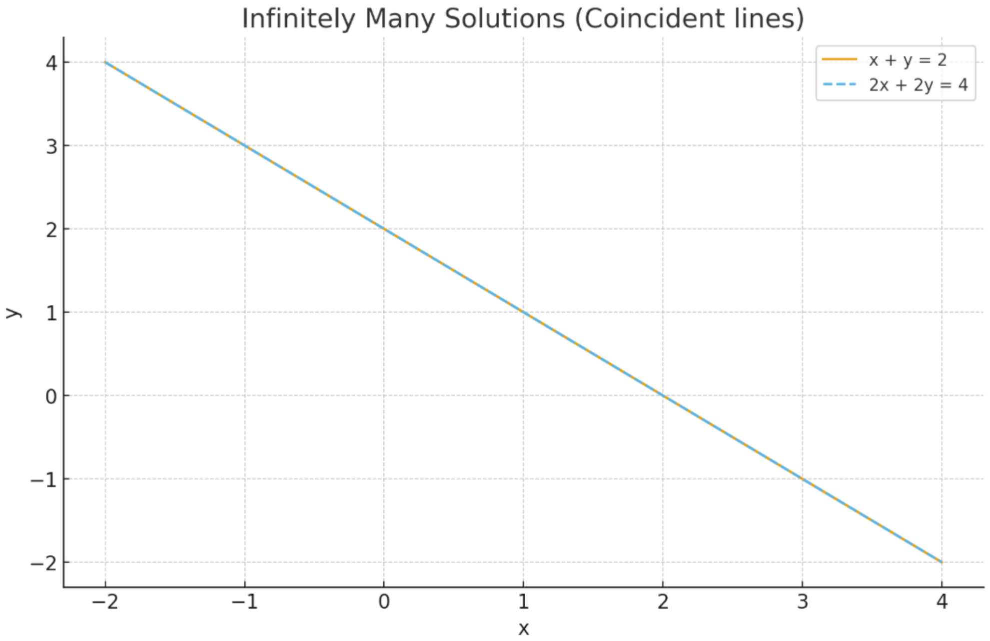
- Then draw lines for these two systems:

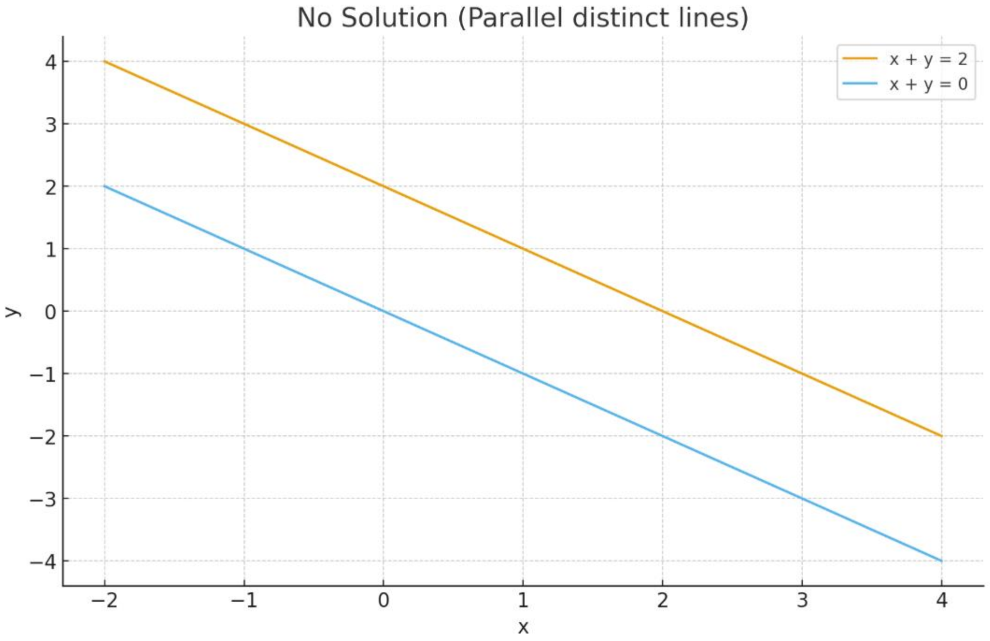
$$\begin{cases} x + y = 2 \\ 2x + 2y = 4 \end{cases} \qquad \begin{cases} x + y = 2 \\ x + y = 0 \end{cases}$$

- What did you find?

Geometric view of these three systems





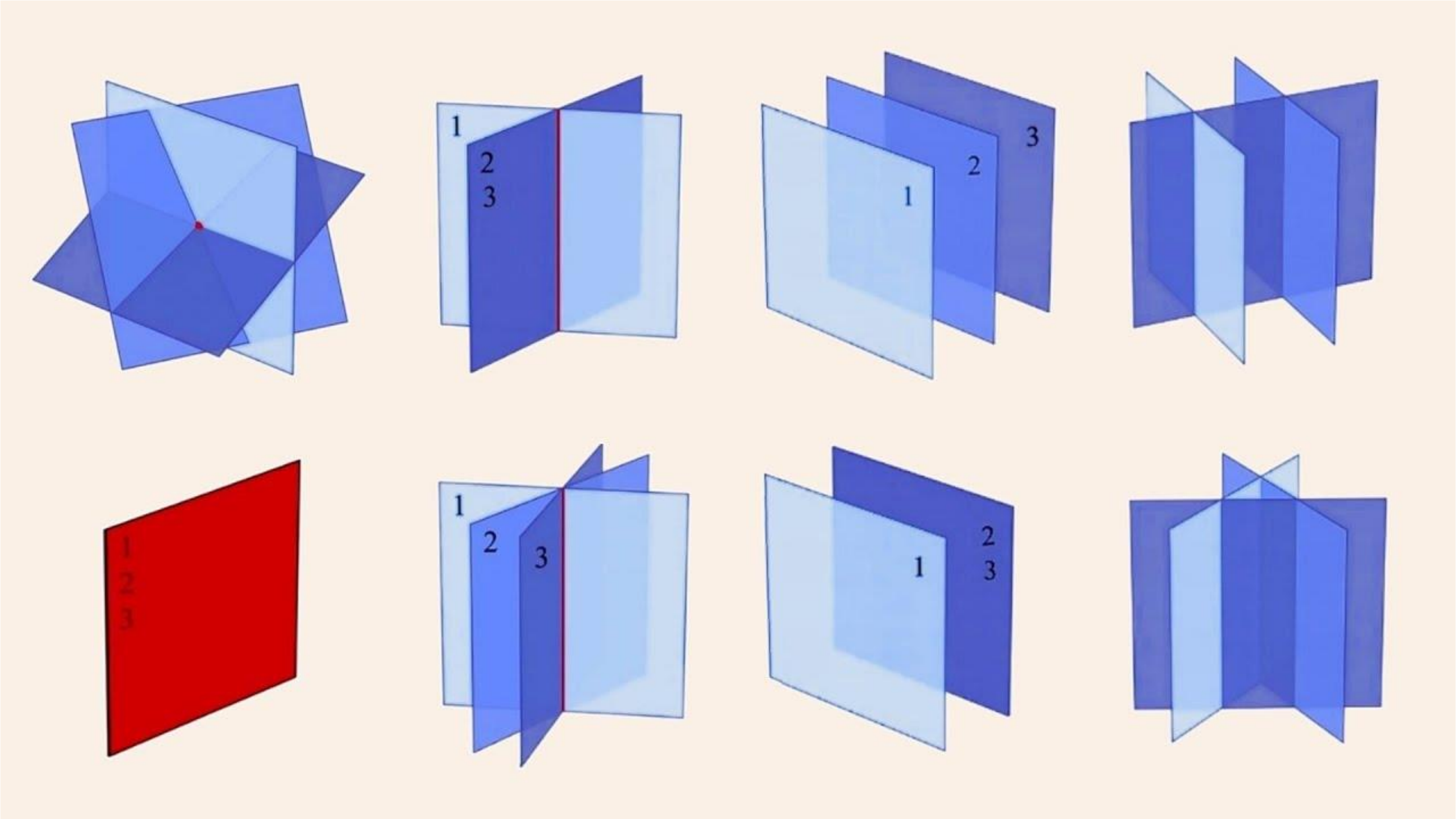


Geometric view of linear systems

- Each linear equation in the system defines a hyperplane.
  - In 2-d space (only x,y), it's a line.
  - In 3-d space (only x,y,z), it's a plane.
  - In high-d space (x,y,z,...), it's a high-dimensional plane.

Geometric view of solutions (3-d case)

- Discussion:
  - Which figure shows a *unique* solution?
  - Which figure shows *infinitely many* solutions?
  - Which figure shows *no* solution?



Summary of Conditions for Solutions of a Linear System $A\mathrm{x} = \mathrm{b}$

Case	Rank Condition	Number of Solutions	Geometric Interpretation
No Solution	$\text{rank}(A) < \text{rank}([A \mid \mathbf{b}])$	None (inconsistent system)	Hyperplanes do not intersect (contradictory equations)
Unique Solution	$\text{rank}(A) = \text{rank}([A \mid \mathbf{b}]) = n$	Exactly **one**	Hyperplanes intersect at a single point
Infinitely Many Solutions	$\text{rank}(A) = \text{rank}([A \mid \mathbf{b}]) < n$	Infinitely many	Hyperplanes intersect along a line, plane, or higher-dimensional subspace