UNIVERSITY AT ALBANY
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CSI 401 (Fall 2025)

Numerical Methods

Lecture 7: Iterative Linear Solvers: Jacobi & Gauss-Seidel

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Back to linear systems

- An example of linear systems
  - Any linear system can always be rewritten in matrix form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}
\qquad
\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
$$

- More generally, $Ax = b$
  - A is an $m \times n$ matrix
  - x is an n-dimensional vector
  - b is an m-dimensional vector
- Problem: given A and b, how can you solve x?

Direct linear system solvers

- Gaussian elimination

$$\begin{bmatrix} 1 & 1 & 1 & 7 \\ 3 & 2 & 1 & 11 \\ 4 & -2 & 2 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & -1 & -2 & -10 \\ 0 & 0 & 10 & 40 \end{bmatrix}$$

- Gauss-Jordan Elimination

- LU decomposition

Agenda

- Iterative linear solvers
  - Jacobi method
  - Gauss-Seidel method

Problem setup

- We want to solve:
- $Ax = b,$
  - where $A \in \mathbb{R}^{n\times n}$,$x, b \in \mathbb{R}^n$.

- Split A into three parts:
- $A = D + L + U,$
  - D: diagonal of A, so $D = \text{diag}(a_{11}, a_{22}, \dots, a_{nn})$
  - L: strictly lower triangular part of A
  - U: strictly upper triangular part of A

An example of D, L, U decomposition

- Linear system:

$$\begin{cases} 4x_1 + x_2 + 2x_3 = 4, \\ x_1 + 3x_2 + x_3 = 5, \\ 2x_1 + x_2 + 5x_3 = 6. \end{cases}$$

- Then,

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 5 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Jacobi method

- After DLU decomposition, we have
  - $(D + L + U)x = b$
- Rearranging gives:
  - $Dx = b - (L + U)x$
- Jacobi iteration updates:
  - $x^{(k+1)} = D^{-1}(b - (L + U)x^{(k)})$.
  - Or equivalently:
    - $x^{(k+1)} = D^{-1}b + \underbrace{(-D^{-1}(L + U))}_{=:\,T_J} x^{(k)}$.
- So in compact form:
  - $x^{(k+1)} = T_J x^{(k)} + c, where\ c = D^{-1}b$.

Jacobi method

- Start with an all-zero vector $x^{(0)} = 0$
- Run for K iterations:
  - $x^{(k+1)} = T_J x^{(k)} + c$
  - where $T_J = -D^{-1}(L + U), c = D^{-1}b.$

- Finally, $x^{(K)}$ will be your approximated solution.

Back to the example

- Problem:

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 5 \end{bmatrix}, \qquad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- In-class exercise:
  - Find T_J and c.

$$T_J = -D^{-1}(L+U) = \begin{bmatrix} 0 & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{2}{5} & -\frac{1}{5} & 0 \end{bmatrix}$$

$$c_J = D^{-1}b = \begin{bmatrix} 1 \\ \frac{5}{3} \\ \frac{6}{5} \end{bmatrix}$$

Two iterations of Jacobi method

- In-class exercise:
  - Work out the first iteration.

$$x^{(1)} = T_J x^{(0)} + c_J = c_J = \begin{bmatrix} 1 \\ \frac{5}{3} \\ \frac{6}{5} \end{bmatrix} \approx \begin{bmatrix} 1.0000 \\ 1.6667 \\ 1.2000 \end{bmatrix}$$

- The second iteration is:

$$x^{(2)} = T_J x^{(1)} + c_J = T_J c_J + c_J.$$

$$T_J c_J = \begin{bmatrix} 0 \cdot 1 - \frac{1}{4} \cdot \frac{5}{3} - \frac{1}{2} \cdot \frac{6}{5} \\ -\frac{1}{3} \cdot 1 + 0 \cdot \frac{5}{3} - \frac{1}{3} \cdot \frac{6}{5} \\ -\frac{2}{5} \cdot 1 - \frac{1}{5} \cdot \frac{5}{3} + 0 \cdot \frac{6}{5} \end{bmatrix} = \begin{bmatrix} -\frac{61}{60} \\ -\frac{11}{15} \\ -\frac{11}{15} \end{bmatrix}$$

$$x^{(2)} = \begin{bmatrix} -\frac{61}{60} + 1 \\ -\frac{11}{15} + \frac{5}{3} \\ -\frac{11}{15} + \frac{6}{5} \end{bmatrix} = \begin{bmatrix} -\frac{1}{60} \\ \frac{14}{15} \\ \frac{7}{15} \end{bmatrix} \approx \begin{bmatrix} -0.0167 \\ 0.9333 \\ 0.4667 \end{bmatrix}$$

Gauss-Seidel method

- After DLU decomposition, we have
  - $(D + L + U)x = b$
- Rearranging gives:
  - $(D + L)x = b - Ux$
- Gass-Seidel iteration updates:
  - $(D + L)x^{(k+1)} = b - Ux^{(k)}$
  - Formally, $x^{(k+1)} = T_{GS}x^{(k)} + c_{GS}$
    - where $T_{GS} = -(D + L)^{-1}U, c_{GS} = (D + L)^{-1}b$

Back to the example

- Problem:

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 5 \end{bmatrix}, \qquad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- In-class exercise:
  - Find T_{GS} and c_{GS} using Python/Matlab.

$$T_{\mathrm{GS}} = -(D+L)^{-1}U = \begin{bmatrix} 0 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & \frac{1}{12} & -\frac{1}{6} \\ 0 & \frac{1}{12} & \frac{7}{30} \end{bmatrix}$$

$$c_{\mathrm{GS}} = \begin{bmatrix} 1 \\ \frac{4}{3} \\ \frac{8}{15} \end{bmatrix}$$

Two iterations of Gauss-Seidel

- In-class exercise:
  - Find the output of the first iteration with $x^{(0)} = 0$.

$$x^{(1)} = c_{GS} = (D+L)^{-1}b = \begin{bmatrix} 1 \\ \frac{4}{3} \\ \frac{8}{15} \end{bmatrix} \approx \begin{bmatrix} 1.0000 \\ 1.3333 \\ 0.5333 \end{bmatrix}$$

- The second iteration is

$$x^{(2)} = T_{GS}\, x^{(1)} + c_{GS} = \begin{bmatrix} \frac{2}{5} \\ \frac{61}{45} \\ \frac{173}{225} \end{bmatrix} \approx \begin{bmatrix} 0.4000 \\ 1.3556 \\ 0.7689 \end{bmatrix}$$

Summary of Jacobi & Gauss-Seidel method

- Both use the L, U, D decomposition
  - $A = D + L + U,$
    - D: diagonal of A
    - L: strictly lower triangular part of A
    - U: strictly upper triangular part of A
- Both are guaranteed to converge if A is symmetric positive definite (SPD).
  - SPD: $A = A^T$ and $x^T Ax > 0$ if any $x \neq 0$.

Recap: In class exercise: prove $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ is a positive definite matrix

- Solution 1: prove $x^T A x \geq 0$ for any vector x.

- Solution 2: prove all eigenvalues of A are all non-negative.
  - Hint: solve $\det(A - \lambda I) = 0$ to find eigenvalues.