UNIVERSITY AT ALBANY
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CSI 401 (Fall 2025)

Numerical Methods

Lecture 6: Eigenvalues & Eigenvectors: Power Method

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Recap: Linear systems

- An example of linear systems
  - Any linear system can always be rewritten in matrix form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}
\qquad
\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
$$

- More generally, $Ax = b$
  - A is a $n \times n$ matrix, x, b are n-dimensional vectors.
- Problem: given A and b, how can you solve x?

Recap: Gaussian elimination

- Rewrite the problem in augmented matrix form
  - Original augmented matrix and manipulated augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 7 \\ 3 & 2 & 1 & 11 \\ 4 & -2 & 2 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & -1 & -2 & -10 \\ 0 & 0 & 10 & 40 \end{bmatrix}$$

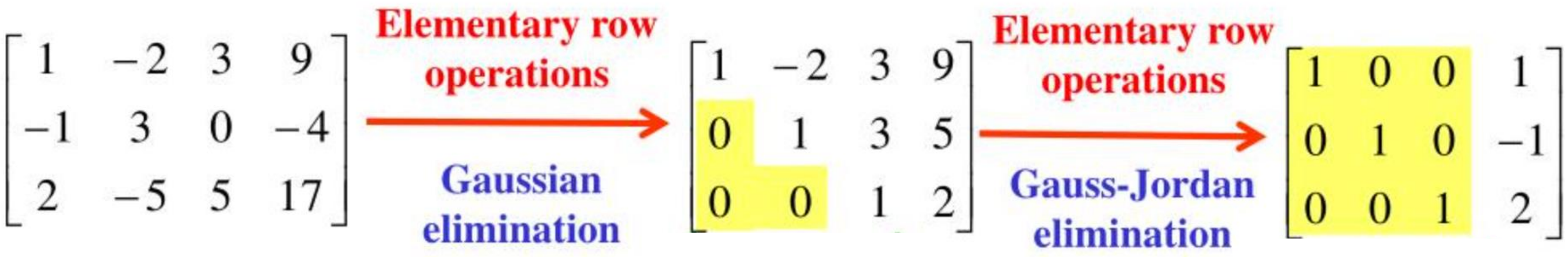
- Elementary row operations:
  - 2^{nd} line = 2^{nd} line – 3 * 1^{st} line [0 -1 -2 -10], done!
  - 3^{rd} line = 3^{rd} line – 4 * 1^{st} line [0 -6 -2 -20]
  - 3^{rd} line = 3^{rd} line – 6 * 2^{nd} line [0 0 10 40], done!

Recap: Gauss-Jordan Elimination

- Consider this linear system:

$$
\begin{aligned}
x - 2y + 3z &= 9 \\
-x + 3y \quad &= -4 \\
2x - 5y + 5z &= 17
\end{aligned}
$$

- Key idea of Gauss-Jordan Elimination:
  - Use elementary row operations to make A become an identity matrix
  - So that you can directly read the results!



Recap: LU decomposition

- $A = LU$
  - L is a lower triangular matrix, U is a upper triangular matrix
  - Note this decomposition may **not** be unique
- For example, LU decomposition of A

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

- To solve linear systems:
  - $Ax = b$ becomes $LUx = b$
  - Solution:
    - Step 1: Solve y from $Ly = b$
    - Step 2: Solve x from $Ux = y$

Partial pivoting prevents unstable solutions

- How does partial pivoting work?
  - Swap rows to make pivot have the largest absolute value in its column.

$$\left(\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 1 & 1 & 2 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 10^{-20} & 1 & 1 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 - 10^{-20} R_1} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 - 10^{-20} & 1 - 2 \cdot 10^{-20} \end{array}\right)$$

$$x_2 = \frac{1 - 2 \cdot 10^{-20}}{1 - 10^{-20}} \approx 1 \pm 10^{-20}$$

$$x_1 + x_2 = 2 \implies x_1 = 1 \pm 10^{-20}.$$

- Why does it work?
  - Avoid dividing by tiny numbers, reduces relative error, and makes LU numerically stable for most matrices.

Agenda

- Eigenvalue and eigenvectors

- Power method

  - Find eigenvalues and eigenvectors

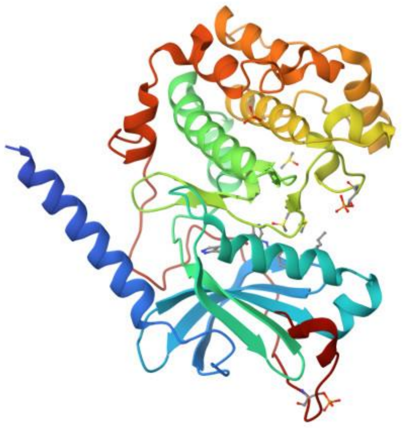
Eigenvalues and eigenvectors

- Definition:
  - For an $n \times n$ matrix A, an eigenvector v of A is a nonzero vector such that there exists some $\lambda \in R$ satisfying
  - $Av = \lambda v.$

- Discussion:
  - What's the dimension of v?
  - Is Av a matrix or a vector?
  - Is λv a vector or a real number?

Applications of Eigenvalues and eigenvectors

- Principal component analysis
  - One of the most widely used dimension-reduction method
  - For
    - Efficient data storage
    - Protein sequence analysis
    - …



- Quantum computing

- And a lot more!

Goal: Find eigenvectors and eigenvalues of a square matrix A

- Discussion: How to find eigenvalues by hand?
  - Reviewed in Lecture 3.

- Work with the characteristic equation for the eigenvalues λ:

$$Av = \lambda v \Leftrightarrow Av - \lambda v = 0 \Leftrightarrow Av - \lambda I v = 0 \Leftrightarrow (A - \lambda I)v = 0.$$

- But v is a nonzero vector, so $\det(A - \lambda I) = 0$.

Detour: How to understand matrix determinant $\det(A)$?

- Intuitively, the determinant measures the **scaling factor** of the linear transformation defined by the matrix.

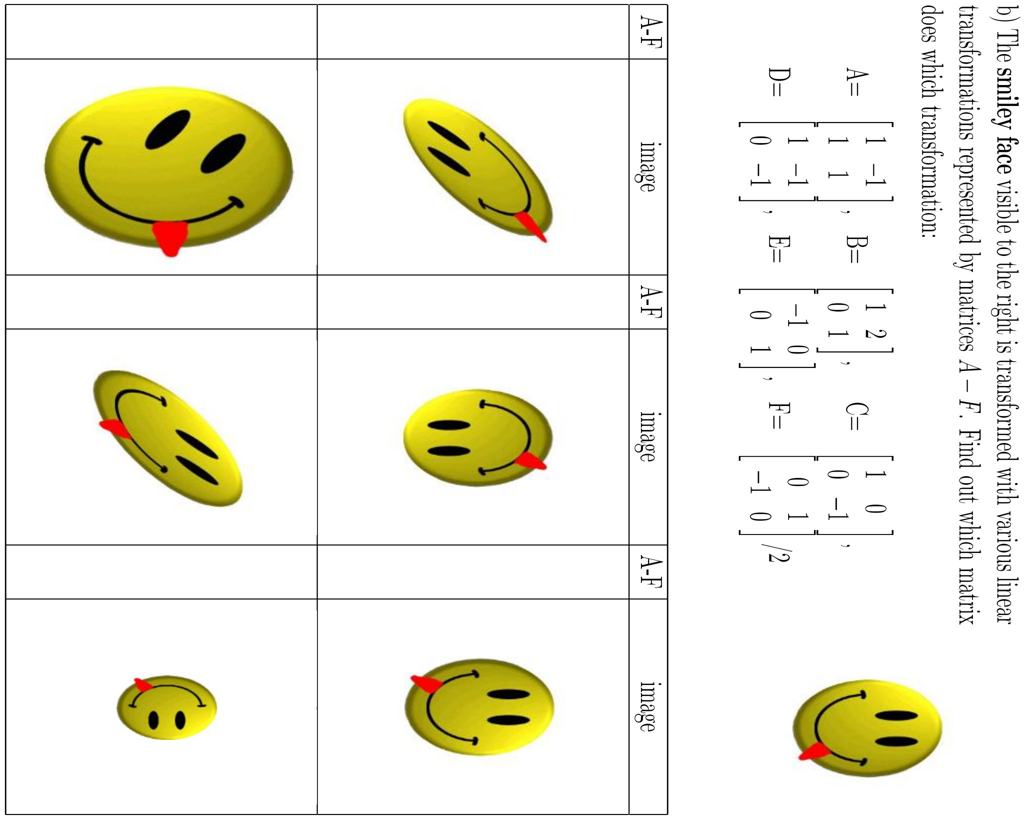
For a **2×2 matrix**

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the determinant is

$$\det(A) = ad - bc$$

- In class exercise:
  - Find det of A, B, C, D, E, F.
  - Discussion: what did you find?



Back to our goal: Find eigenvectors and eigenvalues of a square matrix A

- Find eigenvalues by hand: Work with the characteristic equation for the eigenvalues λ:

$$Av = \lambda v \Leftrightarrow Av - \lambda v = 0 \Leftrightarrow Av - \lambda Iv = 0 \Leftrightarrow (A - \lambda I)v = 0.$$

- But v is a nonzero vector, so $\det(A - \lambda I) = 0$.

- In class exercise: Find eigenvalue and eigenvectors.

$$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$$

Goal: Find eigenvectors and eigenvalues of a square matrix A

- Work with the characteristic equation for the eigenvalues λ:

$$Av = \lambda v \Leftrightarrow Av - \lambda v = 0 \Leftrightarrow Av - \lambda I v = 0 \Leftrightarrow (A - \lambda I)v = 0.$$

- For large matrix A, we cannot find it by hand writing
- So we need an efficient algorithm

- Discussion: Is this a linear system?

Goal: Find eigenvectors and eigenvalues of a square matrix A

- So we want to solve:

$$(A - \lambda I)v = 0.$$

- If A has n linearly independent eigenvectors $v_1, \dots, v_n$, so they collectively form a basis for R^n, then A is diagonalizable

$$A = VDV^{-1}$$

  - V is the matrix whose columns are $v_1, \dots, v_n$, and D is a diagonal matrix whose diagonal entries are $\lambda_1, \dots, \lambda_n$

$$AV = \begin{pmatrix} | & & | \\ Av_1 & \cdots & Av_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \lambda_1 v_1 & \cdots & \lambda_n v_n \\ | & & | \end{pmatrix} = VD.$$

If A has n linearly independent eigenvectors $v_1, \ldots, v_n$, so they collectively form a basis for R^n, then A is diagonalizable

- Let $v_1, \ldots, v_n$ be orthonormal eigenvectors of A. Then any vector x can be written as

$$x = v_1^T x \cdot v_1 + \ldots v_n^T x \cdot v_n.$$

  - That is: x can be written as a sum of its orthogonal projections onto each of the vectors.

- Then Ax is simply

$$Ax = \lambda_1 v_1^T x \cdot v_1 + \ldots \lambda_n v_n^T x \cdot v_n.$$

Power method: Computing the eigenvalue of largest modulus and its corresponding eigenvector

$$|\lambda_1| > |\lambda_2| \geqslant \ldots \geqslant |\lambda_n|.$$

- Works for diagonalizable matrix only. All symmetric matrices are diagonalizable.

- Algorithm:
  - Start with an initial nonzero vector $w^{(0)}$
  - Run in K iterations

$$w^{(k+1)} = \frac{Aw^{(k)}}{\|Aw^{(k)}\|_2}.$$

  - Then your final $w^{(K)} \approx v_1$
  - And $\lambda_1 = Av_1/v_1$

Derivation of the Power Method

$$w^{(0)} = c_1 v_1 + c_2 v_2 + ... + c_n v_n,$$

for coefficients $c_1, ..., c_n$. Then

$$\begin{aligned} A^k w^{(0)} &= c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2 + ... + c_n \lambda_n^k v_n \\ &= \lambda_1^k \cdot (c_1 v_1 + c_2 (\lambda_2/\lambda_1)^k v_2 + ... + c_n (\lambda_n/\lambda_1)^k v_n). \end{aligned}$$

Note that since $|\lambda_j| < |\lambda_1|$ for all j, we have

$$(\lambda_j/\lambda_1)^k \xrightarrow{k \to \infty} 0.$$

Thus,

$$\frac{1}{\lambda_1^k} A^k w^{(0)} \to c_1 v_1.$$

Appropriate normalization yields v_1. In particular, it can be shown that

$$\frac{A^k w^{(0)}}{\|A^k w^{(0)}\|_2} = w^{(k+1)}.$$

Power method in MATLAB

```
function [v, lambda] = power_method(A, w, k)
    for j=1:k
        w = A*w / norm(A*w);
    end
    v = w;
    z = A*w;
    lambda = z(1) / v(1);
end
```

- Discussion: stopping criterion is number of iterations. What else can be a stopping criterion?

In-class exercise: 2 iterations of power method

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

whose eigenvalues are 3 and 1 (dominant $= 3$, eigenvector $[1, 1]^T$).

Start $x^{(0)} = (1, 0)^T$.

- Solution:
  - $Ax^{(0)} = [2; 1], x^{(1)} = \frac{[2;1]}{\sqrt{5}}, Ax^{(1)} = \frac{[5;3]}{\sqrt{5}}, x^{(2)} = \frac{[5;3]}{\sqrt{\frac{34}{5}}}$

An application of power method

Example 15.6 (A successful application of the power method). *Consider the matrix*

$$\begin{pmatrix} 1 & 2 & 1 \\ -4 & 7 & 1 \\ -1 & -2 & -1 \end{pmatrix} \tag{15.21}$$

Let us find its dominant eigenvalue and eigenvector using the power method. We will start with the vector

$$w^{(0)} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \tag{15.22}$$

We eventually find

$$v_1 \approx w^{(k)} = \begin{pmatrix} 0.348155311911396 \\ 0.870388279778489 \\ -0.348155311911396 \end{pmatrix} \tag{15.23}$$

Let us find the corresponding eigenvalue.

$$z = Av_1 = \begin{pmatrix} 1.740776559556978 \\ 4.351941398892445 \\ -1.740776559556978 \end{pmatrix} \tag{15.24}$$

We can compute $z_1/v_{1,1}$*, which yields* $\lambda_1 = 5$.

Summary

- Power method is to used to calculate eigenvalue and eigenvector of a matrix
  - In an iterative way
    - Stopping criterion: number of iterations, relative error
  - Works for diagonalizable matrices only
    - All symmetric matrices are diagonalizable
  - Only finds the eigenvalue of the largest absolute value and its associated eigenvector
    - HW2 also requires you to find the second largest eigenvalue. How?